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THE CAUCHY DEFINITION OF A DEFINITE INTEGRAL.

By D. C. Gillespie.

Cauchy* gave the following definition of the definite integral of a *continuous* function $f(x)$ between the limits (a, b):

$$\int_a^b f(x)dx = \text{Lim}\,[f(a)(x_1-a)+f(x_1)(x_2-x_1)+\cdots+f(x_{m-1})(b-x_{m-1})]. \quad (1)$$

In† justifying his definition he proves that the limit of the sum on the right is the same for all modes of subdivisions of (a, b) in which the limit of the largest sub-interval is zero. He also proves of the integral $\int_a^b f(x)dx$ as thus defined that

$$\int_a^b f(x)dx = \text{Lim}\,[f(\xi_1)(x_1-a)+f(\xi_2)(x_2-x_1)+\cdots+f(\xi_m)(b-x_{m-1})], \quad (2)$$

where $\xi_1\cdots\xi_m$ denote numbers chosen at random in $(a, x_1)\cdots(x_{m-1}, b)$. Riemann‡ takes the limit of the sum (2) as his definition of the definite integral of any function $f(x)$ in the interval (a, b).

A bounded function $f(x)$ will therefore be said to be integrable in the Cauchy sense if the limit on the right in (1) is unique for all modes of subdivision of the interval (a, b) in which the limit of the largest sub-interval is zero; and in the Riemann sense if the like is true of the limit on the right in (2). It is the object of this note to prove that these two definitions are equivalent.

Since the sum (1) is included among the sums (2), if $f(x)$ is integrable in the Riemann sense it is obviously integrable in the Cauchy sense. It is therefore only necessary to prove that if $f(x)$ is not integrable in the Riemann sense it is not integrable in the Cauchy sense.

The necessary§ and sufficient condition that $f(x)$ be integrable in the Riemann sense is that every closed set‖ contained in the set of points at which the oscillation** of $f(x)$ is greater than any positive number k that may be assigned has the content* zero. Hence if $f(x)$ is not integrable in the Riemann sense there is at least one closed set G of content $c(> 0)$ at the points of which the oscillations of $f(x)$ are greater than some constant $k(> 0)$. But it can be shown as follows that if such a set G exists, two systems of subdivisions of (a, b), D_1 and D_2, can be formed for which the limits, granted that they exist, of the sums of the form (1) will differ by $ck/8$; so that $f(x)$ is not integrable in the Cauchy sense.

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* Cauchy, Léçons sur le Calcul Infinitesimal, p. 81.

† Cauchy, Oeuvres, Ser. 2, Vol. IV, p. 122–128.

‡ Riemann, Werke, p. 213.

§ Hobson, Theory of Functions of a Real Variable, p. 342.

‖ Hobson, loc. cit., p. 64.

** Oscillation-Saltus, Hobson, loc. cit., p. 233.

Let D be a division of the interval (a, b) into m equal subintervals. To prove the existence of D_1 and D_2 we shall consider those intervals of division D which contain points of G either within or at the end points of the intervals. Let x_i, x_{i+1} be the end points of such an interval. If x_i is not a point of G, then there is a first point of G beyond x_i, since G is a closed set. Let us designate this first point of G, provided it lies in the interior of the interval (x_i, x_{i+1}), by p_i. If x_i is a point of G take a point v_i such that $x_i < v_i < x_{i+1}$ and such that $(v_i - x_i) < c/2m$. If v_i is a point of G we shall also designate it by p_i. If v_i is not a point of G, and there are points of G between v_i and x_{i+1}, we shall designate the first of these points beyond v_i by p_i. Apply this process to all the intervals of the division D which contain points of G. Then all the points of G, except possibly b, will lie in a set of intervals, not greater than $2m$ in number, such as (x_i, v_i) and (p_i, x_{i+1}). Each of the intervals (x_i, v_i) is less than $c/2m$; since there are not more than m of them, their sum is less than $c/2$. Therefore the sum of the intervals (p_i, x_{i+1}) must be greater than $c/2$.

We choose now a positive number d_i smaller than the smallest of the three numbers $(p_i - x_i)$, $(x_{i+1} - p_i)$ and $ck/32mM$ (where M is the upper limit of $|f(x)|$ in (a, b)). Then with p_i as center construct an interval of total length $2d_i$. Within this interval about p_i there are two points, which we shall designate by s_i and t_i, such that $f(s_i) - f(t_i) > k/2$.

Let the division D_1 have for end points all the end points of D and in addition the points s_i, and let the division D_2 have for end points all the end points of D and in addition the points t_i. Now the sum (1) due to the division D_1 minus the sum (1) due to the division D_2 is greater than $ck/8$. For the sum (1) due to division D_1 would be made up of terms such as $f(x_i)(s_i - x_i) + f(s_i)(x_{i+1} - s_i)$, together with terms due to those intervals of D in which there is no point p_i. The sum (1) due to division D_2 would be made up of terms such as $f(x_i)(t_i - x_i) + f(t_i)(x_{i+1} - t_i)$, together with terms due to those intervals of D in which there is no point p_i. The difference between the terms of D_1 and D_2 just indicated is

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* Hobson, loc. cit., p. 98. Content, as used here, is called upper content by Pierpont, Theory of Functions of Real Variables, p. 352.

$$f(x_i)(s_i - t_i) + f(s_i)(x_{i+1} - p_i + p_i - s_i) - f(t_i)(x_{i+1} - p_i + p_i - t_i)$$
$$= f(x_i)(s_i - t_i) + (f(s_i) - f(t_i))(x_{i+1} - p_i) + f(s_i)(p_i - s_i) - f(t_i)(p_i - t_i).$$

The total difference between the sum (1) due to division D_1 and the sum (1) due to division D_2 would be the sum of all such terms.

$$| \Sigma f(x_i)(s_i - t_i) | < \frac{ck}{16}, \text{ since } | f(x_i) \leqq M \text{ and } |(s_i - t_i)| < 2\delta i < \frac{ck}{16Mm}.$$

$$\Sigma(f(s_i) - f(t_i))(x_{i+1} - p_i) > \frac{ck}{4}, \text{ since } f(s_i) - f(t_i) > \frac{k}{2} \text{ and } \Sigma(x_{i+1} - p_i) > \frac{c}{2}.$$

$$| \Sigma(f(s_i)(p_i - s_i) | < \frac{ck}{32}, \text{ since } |f(s_i)| \leqq M \text{ and } | p_i - s_i | < \delta_i < \frac{ck}{32Mm}.$$

$$| \Sigma(f(t_i)(p_i - t_i) | < \frac{ck}{32}, \text{ since } |f(t_i)| \leqq M \text{ and } | p_i - t_i | < \delta_i < \frac{ck}{32Mm}.$$

Therefore the sum due to D_1 minus the sum due to D_2 is greater than $ck/8$. This difference is independent of m. *Thus as m becomes infinite the limits of the sums due to D_1 and D_2 can not both exist and be equal.*

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